Special Relativity: Types of Forces

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In special relativity, this paper presents four net forces, which can be applied in any massive or non-massive particle, and where the relationship between net force and special acceleration is as in Newton's second law (that is, the special acceleration of any massive or non-massive particle is always in the direction of the net force acting on the particle)

Introduction

In special relativity, this paper is obtained starting from the essential definitions of intrinsic mass (or invariant mass) and relativistic factor (or frequency factor) for massive particles and non-massive particles.

The intrinsic mass (m) and the relativistic factor (f) of a massive particle, are given by:

$$m \doteq m_o \tag{1}$$

$$f \doteq \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}\right)^{-1/2} \tag{2}$$

where (m_o) is the rest mass of the massive particle, ($\mathbf{v}$) is the velocity of the massive particle, and (c) is the speed of light in vacuum.

The intrinsic mass (m) and the relativistic factor (f) of a non-massive particle, are given by:

$$m \doteq \frac{h\,\kappa}{c^2} \tag{3}$$

$$f \doteq \frac{\nu}{\kappa} \tag{4}$$

where (h) is the Planck constant, (ν) is the frequency of the non-massive particle, (κ) is a positive universal constant with dimension of frequency, and (c) is the speed of light in vacuum.

According to this paper, a massive particle ($m_o \neq 0$) is a particle with non-zero rest mass (or a particle whose speed v in vacuum is less than c) and a non-massive particle ($m_o = 0$) is a particle with zero rest mass (or a particle whose speed v in vacuum is c)

Note : The rest mass (m_o) and the intrinsic mass (m) are in general not additive, and the relativistic mass (m) of a particle (massive or non-massive) is given by : ($\mathrm{m} \doteq m\,f$)

The Einsteinian Kinematics

The special position ($\bar{\mathbf{r}}$) the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a particle (massive or non-massive) are given by:

$$\bar{\mathbf{r}} \doteq \int f\,\mathbf{v}\,dt \tag{5}$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = f\,\mathbf{v} \tag{6}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v} \tag{7}$$

where (f) is the relativistic factor of the particle, ($\mathbf{v}$) is the velocity of the particle, and (t) is the (coordinate) time.

The Einsteinian Dynamics

If we consider a particle (massive or non-massive) with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net Einsteinian force ($\mathbf{F}_{\mathrm{E}}$) acting on the particle, the work (W) done by the net Einsteinian force acting on the particle, and the kinetic energy (K) of the particle, are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,f\,\mathbf{v} \tag{8}$$

$$\mathbf{L} \doteq \mathbf{r}\times\mathbf{P} = m\,\mathbf{r}\times\bar{\mathbf{v}} = m\,f\,\mathbf{r}\times\mathbf{v} \tag{9}$$

$$\mathbf{F}_{\mathrm{E}} = \frac{d\mathbf{P}}{dt} = m\,\bar{\mathbf{a}} = m\left[f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}\right] \tag{10}$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F}_{\mathrm{E}}\cdot d\mathbf{r} = \int_1^2 \frac{d\mathbf{P}}{dt}\cdot d\mathbf{r} = \Delta\,\mathrm{K} \tag{11}$$

$$\mathrm{K} \doteq m\,f\,c^2 \tag{12}$$

where (f, $\mathbf{r}$, $\mathbf{v}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the relativistic factor, the position, the velocity, the special velocity and the special acceleration of the particle, (t) is the (coordinate) time, and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at rest is ($m_o\,c^2$) since in this dynamics the relativistic energy ($\mathrm{E} \doteq m_o\,c^2\,(\,f-1\,) + m_o\,c^2$) and the kinetic energy ($\mathrm{K} \doteq m\,f\,c^2$) are the same ($\mathrm{E} = \mathrm{K}$) [1]

Note : $\mathrm{E}^2 - \mathbf{P}^2 c^2 = m^2\,f^2\,c^4\,(1-\mathbf{v}^2/c^2\,)$ [in massive particle : $f^2\,(1-\mathbf{v}^2/c^2\,) = 1 \;\rightarrow\; \mathrm{E}^2 - \mathbf{P}^2 c^2 = {m_o}^2 c^4$ and $m \neq 0$] & [in non-massive particle : $\mathbf{v}^2 = c^2 \;\rightarrow\; (\,1-\mathbf{v}^2/c^2\,) = 0 \;\rightarrow\; \mathrm{E}^2 - \mathbf{P}^2 c^2 = 0$ and $m \neq 0$]
In special relativity there are 3 types of masses: rest mass (m_o) intrinsic mass (m) and relativistic mass (m)

The Newtonian Kinematics

The special position ($\bar{\mathbf{r}}$) the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a particle (massive or non-massive) are given by:

$$\bar{\mathbf{r}} \doteq \mathbf{r} \tag{13}$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = \mathbf{v} \tag{14}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = \mathbf{a} \tag{15}$$

where ($\mathbf{r}$) is the position of the particle, ($\mathbf{v}$) is the velocity of the particle, ($\mathbf{a}$) is the acceleration of the particle, and (t) is the (coordinate) time.

The Newtonian Dynamics

If we consider a particle (massive or non-massive) with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net Newtonian force ($\mathbf{F}_{\mathrm{N}}$) acting on the particle, the work (W) done by the net Newtonian force acting on the particle, and the kinetic energy (K) of the particle, are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,\mathbf{v} \tag{16}$$

$$\mathbf{L} \doteq \mathbf{r} \times \mathbf{P} = m\,\mathbf{r} \times \bar{\mathbf{v}} = m\,\mathbf{r} \times \mathbf{v} \tag{17}$$

$$\mathbf{F}_{\mathrm{N}} = \frac{d\mathbf{P}}{dt} = m\,\bar{\mathbf{a}} = m\,\mathbf{a} \tag{18}$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F}_{\mathrm{N}} \cdot d\mathbf{r} = \int_1^2 \frac{d\mathbf{P}}{dt} \cdot d\mathbf{r} = \Delta\,\mathrm{K} \tag{19}$$

$$\mathrm{K} \doteq {}^1\!/_2\; m\,(\mathbf{v} \cdot \mathbf{v}) \tag{20}$$

where ($\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the position, the velocity, the acceleration, the special velocity and the special acceleration of the particle, (t) is the (coordinate) time, and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at rest is zero, and the ordinary acceleration ($\mathbf{a}$) of a massive or non-massive particle is also always in the direction of the net Newtonian force ($\mathbf{F}_{\mathrm{N}}$) acting on the particle.

In special relativity, the net Newtonian force ($\mathbf{F}_{\mathrm{N}}$) acting on a massive or non-massive particle, is given by : $\mathbf{F}_{\mathrm{N}} \doteq \mathbf{N}^{-1} \cdot \mathbf{F}_{\mathrm{E}}$, where ($\mathbf{N}$) is the Newton tensor, and ($\mathbf{F}_{\mathrm{E}}$) is the net Einsteinian force acting on the massive or non-massive particle [2]

The Poincarian Kinematics

The special position ($\bar{\mathbf{r}}$) the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a particle (massive or non-massive) are given by:

$$\bar{\mathbf{r}} \doteq \mathbf{r} \tag{21}$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{d\tau} = f\,\mathbf{v} \tag{22}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{d\tau} = f\left[f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}\right] \tag{23}$$

where (f) is the relativistic factor of the particle, ($\mathbf{r}$) is the position of the particle, ($\mathbf{v}$) is the velocity of the particle, and (τ) is the proper time of the particle (Note : $d\tau = f^{-1}\,dt$)

The Poincarian Dynamics

If we consider a particle (massive or non-massive) with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net Poincarian force ($\mathbf{F}_{\mathrm{P}}$) acting on the particle, the work (W) done by the net Poincarian force acting on the particle, and the kinetic energy (K) of the particle, are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,f\,\mathbf{v} \tag{24}$$

$$\mathbf{L} \doteq \mathbf{r}\times\mathbf{P} = m\,\mathbf{r}\times\bar{\mathbf{v}} = m\,f\,\mathbf{r}\times\mathbf{v} \tag{25}$$

$$\mathbf{F}_{\mathrm{P}} = \frac{d\mathbf{P}}{d\tau} = m\,\bar{\mathbf{a}} = m\,f\left[f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}\right] \tag{26}$$

$$\mathrm{W} \doteq \int_1^2 f^{-1}\,\mathbf{F}_{\mathrm{P}}\cdot d\mathbf{r} = \int_1^2 f^{-1}\,\frac{d\mathbf{P}}{d\tau}\cdot d\mathbf{r} = \Delta\,\mathrm{K} \tag{27}$$

$$\mathrm{K} \doteq m\,f\,c^2 \tag{28}$$

where (f, $\mathbf{r}$, $\mathbf{v}$, τ, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the relativistic factor, the position, the velocity, the proper time, the special velocity and the special acceleration of the particle, and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at rest is ($m_o\,c^2$) since also in this dynamics the relativistic energy ($\mathrm{E} \doteq m_o\,c^2\,(\,f-1\,) + m_o\,c^2$) and the kinetic energy ($\mathrm{K} \doteq m\,f\,c^2$) are the same ($\mathrm{E} = \mathrm{K}$)

In special relativity, the net Poincarian force ($\mathbf{F}_{\mathrm{P}}$) acting on a massive or non-massive particle is given by : $\mathbf{F}_{\mathrm{P}} \doteq f\,\mathbf{F}_{\mathrm{E}}$, where (f) is the relativistic factor of the particle, and ($\mathbf{F}_{\mathrm{E}}$) is the net Einsteinian force acting on the massive or non-massive particle [3]

The Møllerian Kinematics

The special position ($\bar{\mathbf{r}}$) the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a particle (massive or non-massive) are given by:

$$\bar{\mathbf{r}} \doteq \int \mathbf{v} \, d\tau \tag{29}$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{d\tau} = \mathbf{v} \tag{30}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{d\tau} = f\,\mathbf{a} \tag{31}$$

where (f) is the relativistic factor of the particle, ($\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$) are the position, the velocity and the acceleration of the particle, and (τ) is the proper time of the particle (Note : $d\tau = f^{-1}\,dt$)

The Møllerian Dynamics

If we consider a particle (massive or non-massive) with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net Møllerian force ($\mathbf{F}_{\mathrm{M}}$) acting on the particle, the work (W) done by the net Møllerian force acting on the particle, and the kinetic energy (K) of the particle, are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,\mathbf{v} \tag{32}$$

$$\mathbf{L} \doteq \mathbf{r} \times \mathbf{P} = m\,\mathbf{r} \times \bar{\mathbf{v}} = m\,\mathbf{r} \times \mathbf{v} \tag{33}$$

$$\mathbf{F}_{\mathrm{M}} = \frac{d\mathbf{P}}{d\tau} = m\,\bar{\mathbf{a}} = m\,f\,\mathbf{a} \tag{34}$$

$$\mathrm{W} \doteq \int_1^2 f^{-1}\,\mathbf{F}_{\mathrm{M}} \cdot d\mathbf{r} = \int_1^2 f^{-1}\,\frac{d\mathbf{P}}{d\tau} \cdot d\mathbf{r} = \Delta\,\mathrm{K} \tag{35}$$

$$\mathrm{K} \doteq {}^1\!/_2\; m\,(\mathbf{v} \cdot \mathbf{v}) \tag{36}$$

where (f, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, τ, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the relativistic factor, the position, the velocity, the acceleration, the proper time, the special velocity and the special acceleration of the particle, and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at rest is zero, and the ordinary acceleration ($\mathbf{a}$) of a massive or non-massive particle is also always in the direction of the net Møllerian force ($\mathbf{F}_{\mathrm{M}}$) acting on the particle.

In special relativity, the net Møllerian force ($\mathbf{F}_{\mathrm{M}}$) acting on a massive or non-massive particle is given by : $\mathbf{F}_{\mathrm{M}} \doteq \mathbf{M} \cdot \mathbf{F}_{\mathrm{E}}$, where ($\mathbf{M}$) is the Møller tensor, and ($\mathbf{F}_{\mathrm{E}}$) is the net Einsteinian force acting on the massive or non-massive particle (see Annex I) [4]

General Observations

In special relativity, the net forces [$\mathbf{F}_{\text{N}}, \mathbf{F}_{\text{P}}, \mathbf{F}_{\text{M}}$] are valid since these net forces are obtained from the net Einsteinian force [$\mathbf{F}_{\text{E}}$]

Therefore, the net forces [$\mathbf{F}_{\text{E}}, \mathbf{F}_{\text{N}}, \mathbf{F}_{\text{P}}, \mathbf{F}_{\text{M}}$] can be applied in any inertial reference frame.

The special acceleration ($\bar{\mathbf{a}}$) of a particle (massive or non-massive) is always in the direction of the net forces [$\mathbf{F}_{\text{E}}, \mathbf{F}_{\text{N}}, \mathbf{F}_{\text{P}}, \mathbf{F}_{\text{M}}$] acting on the particle (as in Newton's second law)

Additionally, the ordinary acceleration ($\mathbf{a}$) of a particle (massive or non-massive) is also always in the direction of the net forces [$\mathbf{F}_{\text{N}}, \mathbf{F}_{\text{M}}$] acting on the particle (exactly as in Newton's second law) (Note : $\mathbf{F}_{\text{N}} = f^{-1}\, \mathbf{F}_{\text{M}} = f^{-1}\, \mathbf{M} \cdot \mathbf{F}_{\text{E}}$)

The net forces [$\mathbf{F}_{\text{E}}, \mathbf{F}_{\text{N}}, \mathbf{F}_{\text{P}}, \mathbf{F}_{\text{M}}$] are three-forces (that is, they are three-dimensional vectors)

On the other hand, the net Minkowskian four-force [$\overline{\mathbf{F}}_{\text{M}}$] is obtained from the four-momentum and the proper time of a massive particle. In addition, the net Einsteinian four-force [$\overline{\mathbf{F}}_{\text{E}}$] can be obtained from the four-momentum and the (coordinate) time of a massive particle (Note : $\overline{\mathbf{F}}_{\text{E}} = (\, (d\text{E}/dt)\, c^{-1} \, , \mathbf{F}_{\text{E}} \,)$ and $\overline{\mathbf{F}}_{\text{M}} = f \; \overline{\mathbf{F}}_{\text{E}}$) [see : Appendix A and Appendix B]

In special relativity, there are three types of masses that are compatible with each other : the rest mass (m_o) the intrinsic mass (m) and the relativistic mass (m) (the intrinsic mass (m) is an invariant mass that can be applied to massive and non-massive particles)

In the Poincarian dynamics, the definition of work (W) is modified so that the magnitudes ($\mathbf{P}, \text{K}$) match the magnitudes ($\mathbf{P}, \text{K}$) of the Einsteinian dynamics.

In the Møllerian dynamics, the definition of work (W) is modified so that the magnitudes ($\mathbf{P}, \text{K}$) match the magnitudes ($\mathbf{P}, \text{K}$) of the Newtonian dynamics.

Additionally, in relativistic elastic collisions (or relativistic elastic shocks) between massive and/or non-massive particles of an isolated system, the magnitudes ($\mathbf{P} = \sum m_i \, f_i \, \mathbf{v}_i$) and ($\text{K} = \sum m_i \, f_i \, c^2$) are conserved [and the net Einsteinian force ($\mathbf{F}_{\text{E}} = d\mathbf{P}/dt$) is always zero]

References & Bibliography

[1] **A. Tobla**, A Reformulation of Special Relativity, (2024). (doi)

[2] **A. Blato**, Special Relativity & Newton's Second Law, (2016). (doi)

[3] **A. Blato**, A New Dynamics in Special Relativity, (2016). (doi)

[4] **C. Møller**, The Theory of Relativity, (1952).

[A] **W. Pauli**, Theory of Relativity, (1958).

[B] **A. French**, Special Relativity, (1968).

Annex I

The Møller Tensor

The Møller tensor ($\mathbf{M}$) and the net Møllerian force ($\mathbf{F}_{\mathrm{M}}$) can be obtained from the net Einsteinian force ($\mathbf{F}_{\mathrm{E}}$) acting on a massive particle with rest mass (m_o)

$$m_o \left[\frac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \frac{(\mathbf{a}\cdot\mathbf{v})\,\mathbf{v}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \right] = \mathbf{F}_{\mathrm{E}} \tag{37}$$

$$m_o \left[\frac{\mathbf{a}\cdot\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \frac{(\mathbf{a}\cdot\mathbf{v})\,(\mathbf{v}\cdot\mathbf{v})}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \right] = \mathbf{F}_{\mathrm{E}}\cdot\mathbf{v} \tag{38}$$

$$m_o \left[\frac{(\mathbf{a}\cdot\mathbf{v})\,\mathbf{v}}{c^2\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \frac{(\mathbf{a}\cdot\mathbf{v})\,(\mathbf{v}\cdot\mathbf{v})\,\mathbf{v}}{c^4\left(1-\frac{v^2}{c^2}\right)^{3/2}} \right] = \frac{(\mathbf{F}_{\mathrm{E}}\cdot\mathbf{v})\,\mathbf{v}}{c^2} \tag{39}$$

$$m_o \left[\frac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \right] = \mathbf{F}_{\mathrm{E}} - \frac{(\mathbf{F}_{\mathrm{E}}\cdot\mathbf{v})\,\mathbf{v}}{c^2} \tag{40}$$

$$m_o \left[\frac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \right] = \mathbf{1}\cdot\mathbf{F}_{\mathrm{E}} - \frac{(\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{F}_{\mathrm{E}}}{c^2} \tag{41}$$

$$m_o \left[\frac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \right] = \left[\mathbf{1} - \frac{(\mathbf{v}\otimes\mathbf{v})}{c^2} \right]\cdot\mathbf{F}_{\mathrm{E}} \tag{42}$$

$$m_o \left[\frac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \right] = \mathbf{M}\cdot\mathbf{F}_{\mathrm{E}} \tag{43}$$

$$m_o \left[\frac{\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \right] = \mathbf{F}_{\mathrm{M}} \tag{44}$$

Note : $\mathbf{F}_{\mathrm{E}} = \mathbf{1}\cdot\mathbf{F}_{\mathrm{E}}$ ($\mathbf{1}$ unit tensor) & $(\mathbf{F}_{\mathrm{E}}\cdot\mathbf{v})\,\mathbf{v} = (\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{F}_{\mathrm{E}}$ ($\otimes$ tensor or dyadic product)

Annex II

The Kinetic Forces

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i with intrinsic mass m_i by another particle j with intrinsic mass m_j, is given by:

$$\mathbf{K}^a_{ij} \;=\; - \left[\; \frac{m_i \, m_j}{\mathbb{M}} \, (\, \bar{\mathbf{a}}_i \, - \, \bar{\mathbf{a}}_j \,) \; \right] \tag{45}$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i, $\bar{\mathbf{a}}_j$ is the special acceleration of particle j and $\mathbb{M}$ ($= \sum_z^{All} m_z$) is the sum of the intrinsic masses of all the particles of the Universe.

On the other hand, the kinetic force $\mathbf{K}^u_i$ exerted on a particle i with intrinsic mass m_i by the Universe, is given by:

$$\mathbf{K}^u_i \;=\; - \, m_i \, \frac{\sum_z^{All} m_z \, \bar{\mathbf{a}}_z}{\sum_z^{All} m_z} \tag{46}$$

where m_z and $\bar{\mathbf{a}}_z$ are the intrinsic mass and the special acceleration of the z-th particle of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_j^{All} \mathbf{K}^a_{ij} + \mathbf{K}^u_i$) acting on a particle i with intrinsic mass m_i, is given by:

$$\mathbf{K}_i \;=\; - \, m_i \, \bar{\mathbf{a}}_i \tag{47}$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i.

Now, from all dynamics $[\,(10), (18), (26), (34)\,]$ we have:

$$\mathbf{F}_i \;=\; m_i \, \bar{\mathbf{a}}_i \tag{48}$$

Since ($\mathbf{K}_i \;=\; - \, m_i \, \bar{\mathbf{a}}_i$) we obtain:

$$\mathbf{F}_i \;=\; - \, \mathbf{K}_i \tag{49}$$

that is:

$$\mathbf{K}_i \, + \, \mathbf{F}_i \;=\; 0 \tag{50}$$

If ($\mathbf{T}_i \doteq \mathbf{K}_i \, + \, \mathbf{F}_i$) then:

$$\mathbf{T}_i \;=\; 0 \tag{51}$$

Therefore, if the net kinetic force $\mathbf{K}_i$ is added in all dynamics then the total force $\mathbf{T}_i$ acting on a (massive or non-massive) particle i is always zero.

Note : According to this paper, the kinetic forces $\overset{au}{\mathbf{K}}$ are directly related to kinetic energy K.

Annex III

System of Particles

In special relativity, the total energy (E) the linear momentum ($\mathbf{P}$) the rest mass (M_o) and the velocity ($\mathbf{V}$) of any massive or non-massive system (of particles) are given by:

$$\mathrm{E} \doteq \sum m_i \, f_i \, c^2 + \sum \mathrm{E}_{\mathrm{nki}} \tag{52}$$

$$\mathbf{P} \doteq \sum m_i \, f_i \, \mathbf{v}_i \tag{53}$$

$$\mathrm{M}_o^2 \, c^4 \doteq \mathrm{E}^2 - \mathbf{P}^2 c^2 \tag{54}$$

$$\mathbf{V} \doteq \mathbf{P} \, c^2 \, \mathrm{E}^{-1} \tag{55}$$

where (m_i, f_i, $\mathbf{v}_i$) are the intrinsic mass, the relativistic factor and the velocity of the i-th massive or non-massive particle of the system, ($\sum \mathrm{E}_{\mathrm{nki}}$) is the total non-kinetic energy of the system, and (c) is the speed of light in vacuum.

The intrinsic mass (M) and the relativistic factor (F) of a massive system (composed of massive particles or non-massive particles, or both at the same time) are given by:

$$\mathrm{M} \doteq \mathrm{M}_o \tag{56}$$

$$\mathrm{F} \doteq \left(1 - \frac{\mathbf{V} \cdot \mathbf{V}}{c^2}\right)^{-1/2} \tag{57}$$

where (M_o) is the rest mass of the massive system, ($\mathbf{V}$) is the velocity of the massive system, and (c) is the speed of light in vacuum.

The intrinsic mass (M) and the relativistic factor (F) of a non-massive system (composed only of non-massive particles, all with the same vector velocity $\mathbf{c}$) are given by:

$$\mathrm{M} \doteq \frac{h \, \kappa}{c^2} \tag{58}$$

$$\mathrm{F} \doteq \frac{1}{\kappa} \sum \nu_i \tag{59}$$

where (h) is the Planck constant, (ν_i) is the frequency of the i-th non-massive particle of the non-massive system, (κ) is a positive universal constant with dimension of frequency, and (c) is the speed of light in vacuum.

According to this paper, a massive system ($\mathrm{M}_o \neq 0$) is a system with non-zero rest mass (or a system whose speed V in vacuum is less than c) and a non-massive system ($\mathrm{M}_o = 0$) is a system with zero rest mass (or a system whose speed V in vacuum is c)

Note : The rest mass (M_o) and the intrinsic mass (M) are in general not additive, and the relativistic mass ($\mathtt{M}$) of a system (massive or non-massive) is given by : ($\mathtt{M} \doteq \mathrm{M}\,\mathrm{F}$)

The Einsteinian Kinematics

The special position ($\bar{\mathbf{R}}$) the special velocity ($\bar{\mathbf{V}}$) and the special acceleration ($\bar{\mathbf{A}}$) of a system (massive or non-massive) are given by:

$$\bar{\mathbf{R}} \doteq \int \mathrm{F}\,\mathbf{V}\,dt \tag{60}$$

$$\bar{\mathbf{V}} \doteq \frac{d\bar{\mathbf{R}}}{dt} = \mathrm{F}\,\mathbf{V} \tag{61}$$

$$\bar{\mathbf{A}} \doteq \frac{d\bar{\mathbf{V}}}{dt} = \mathrm{F}\,\frac{d\mathbf{V}}{dt} + \frac{d\mathrm{F}}{dt}\,\mathbf{V} \tag{62}$$

where (F) is the relativistic factor of the system, ($\mathbf{V}$) is the velocity of the system, and (t) is the (coordinate) time.

The Einsteinian Dynamics

If we consider a system (massive or non-massive) with intrinsic mass (M) then the linear momentum ($\mathbf{P}$) of the system, the angular momentum ($\mathbf{L}$) of the system, the net Einsteinian force ($\mathbf{F}$) acting on the system, the work (W) done by the net Einsteinian forces acting on the system, the kinetic energy (K) of the system, and the total energy (E) of the system, are:

$$\mathbf{P} \doteq \sum \mathbf{p}_i = \sum m_i\,\bar{\mathbf{v}}_i = \sum m_i\,f_i\,\mathbf{v}_i = \mathrm{M}\,\bar{\mathbf{V}} = \mathrm{M}\,\mathrm{F}\,\mathbf{V} \tag{63}$$

$$\mathbf{L} \doteq \sum \mathbf{l}_i = \sum \mathbf{r}_i \times \mathbf{p}_i = \sum m_i\,\mathbf{r}_i \times \bar{\mathbf{v}}_i = \sum m_i\,f_i\,\mathbf{r}_i \times \mathbf{v}_i \tag{64}$$

$$\mathbf{F} = \sum \mathbf{f}_i = \sum \frac{d\mathbf{p}_i}{dt} = \frac{d\mathbf{P}}{dt} = \mathrm{M}\,\bar{\mathbf{A}} = \mathrm{M}\left[\mathrm{F}\,\frac{d\mathbf{V}}{dt} + \frac{d\mathrm{F}}{dt}\,\mathbf{V}\right] \tag{65}$$

$$\mathrm{W} \doteq \sum \int_1^2 \mathbf{f}_i \cdot d\mathbf{r}_i = \sum \int_1^2 \frac{d\mathbf{p}_i}{dt} \cdot d\mathbf{r}_i = \Delta\,\mathrm{K} \tag{66}$$

$$\mathrm{K} \doteq \sum m_i\,f_i\,c^2 \tag{67}$$

$$\mathrm{E} \doteq \sum m_i\,f_i\,c^2 + \sum \mathrm{E_{nki}} = \mathrm{K} + \sum \mathrm{E_{nki}} = \mathrm{M}\,\mathrm{F}\,c^2 \tag{68}$$

where (m_i, f_i, $\mathbf{r}_i$, $\mathbf{v}_i$, $\bar{\mathbf{v}}_i$) are the intrinsic mass, the relativistic factor, the position, the velocity and the special velocity of the i-th massive or non-massive particle of the system, (F, $\mathbf{V}$, $\bar{\mathbf{V}}$, $\bar{\mathbf{A}}$) are the relativistic factor, the velocity, the special velocity and the special acceleration of the system, ($\sum \mathrm{E_{nki}}$) is the total non-kinetic energy of the system, (t) is the (coordinate) time, and (c) is the speed of light in vacuum.

Note : ($\sum \mathrm{E_{nki}} = 0$) in massive or non-massive particle $\rightarrow$ ($\mathrm{E} = \mathrm{K}$) in massive or non-massive particle.

Appendix A

Four-kinematics

The Minkowskian Kinematics

The special four-position (R) the special four-velocity (U) and the special four-acceleration (A) of a particle (massive or non-massive) are given by:

$$\mathsf{R} \doteq \Big(c\,t \ , \ \mathbf{r} \Big) \tag{69}$$

$$\mathsf{U} \doteq \frac{d\mathsf{R}}{d\tau} = \Big(f\,c \ , \ f\,\mathbf{v} \Big) \tag{70}$$

$$\mathsf{A} \doteq \frac{d\mathsf{U}}{d\tau} = f \left(\frac{df}{dt}\,c \ , \ \frac{df}{dt}\,\mathbf{v} + \frac{d\mathbf{v}}{dt}\,f \right) \tag{71}$$

where (f) is the relativistic factor of the particle, ($\mathbf{r}$) is the position of the particle, ($\mathbf{v}$) is the velocity of the particle, and (τ) is the proper time of the particle (Note : $d\tau = f^{-1}\,dt$)

Four-dynamics

The Minkowskian Dynamics

The four-momentum ($\overline{\mathbf{P}}$) of a particle (massive or non-massive) with intrinsic mass (m) and the net Minkowskian four-force ($\overline{\mathbf{F}}_{\mathrm{M}}$) acting on the particle, are given by:

$$\overline{\mathbf{P}} \doteq m\,\mathsf{U} = m \Big(f\,c \ , \ f\,\mathbf{v} \Big) \tag{72}$$

$$\overline{\mathbf{F}}_{\mathrm{M}} = \frac{d\overline{\mathbf{P}}}{d\tau} = m\,\mathsf{A} = m\,f \left(\frac{df}{dt}\,c \ , \ \frac{df}{dt}\,\mathbf{v} + \frac{d\mathbf{v}}{dt}\,f \right) \tag{73}$$

where (f, $\mathbf{v}$, U, A) are the relativistic factor, the velocity, the special four-velocity and the special four-acceleration of the particle, (τ) is the proper time of the particle, and (c) is the speed of light in vacuum.

In the Minkowskian four-mechanics (that is, in the ordinary four-mechanics) all the special four-vectors (R, U, A, $\overline{\mathbf{P}}$, $\overline{\mathbf{F}}_{\mathrm{M}}$) are ordinary four-vectors ($\mathbf{R}$, $\mathbf{U}$, $\mathbf{A}$, $\mathbf{P}$, $\mathbf{F}$).

Additionally, in massive particle : f is the Lorentz factor $\gamma(\mathbf{v})$.

Appendix B

Four-kinematics

The Einsteinian Kinematics

The special four-position (R) the special four-velocity (U) and the special four-acceleration (A) of a particle (massive or non-massive) are given by:

$$\mathsf{R} \doteq \int \left(f\,c \;,\; f\,\mathbf{v} \right) dt \tag{74}$$

$$\mathsf{U} \doteq \frac{d\mathsf{R}}{dt} = \left(f\,c \;,\; f\,\mathbf{v} \right) \tag{75}$$

$$\mathsf{A} \doteq \frac{d\mathsf{U}}{dt} = \left(\frac{df}{dt}\,c \;,\; \frac{df}{dt}\,\mathbf{v} + \frac{d\mathbf{v}}{dt}\,f \right) \tag{76}$$

where (f) is the relativistic factor of the particle, ($\mathbf{v}$) is the velocity of the particle, and (t) is the (coordinate) time.

Four-dynamics

The Einsteinian Dynamics

The four-momentum ($\overline{\mathbf{P}}$) of a particle (massive or non-massive) with intrinsic mass (m) and the net Einsteinian four-force ($\overline{\mathbf{F}}_{\mathrm{E}}$) acting on the particle, are given by:

$$\overline{\mathbf{P}} \doteq m\,\mathsf{U} = m \left(f\,c \;,\; f\,\mathbf{v} \right) \tag{77}$$

$$\overline{\mathbf{F}}_{\mathrm{E}} = \frac{d\overline{\mathbf{P}}}{dt} = m\,\mathsf{A} = m \left(\frac{df}{dt}\,c \;,\; \frac{df}{dt}\,\mathbf{v} + \frac{d\mathbf{v}}{dt}\,f \right) \tag{78}$$

where (f, $\mathbf{v}$, U, A) are the relativistic factor, the velocity, the special four-velocity and the special four-acceleration of the particle, (t) is the (coordinate) time, and (c) is the speed of light in vacuum.

In the Einsteinian four-mechanics, the special four-velocity (U) is the ordinary four-velocity ($\mathbf{U}$) and, therefore, the four-momentum ($\overline{\mathbf{P}}$) is the ordinary four-momentum ($\mathbf{P}$).

Additionally, in massive particle : f is the Lorentz factor $\gamma(\mathbf{v})$.